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An amplitude-evolution equation for linearly unstable modes in stratified shear flows

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A nonlinear amplitude equation of second order in time, which governs the temporal evolution of linearly unstable modes in stratified shear flows, is derived. It applies to a class of flows with continuous velocity and density profiles, and two examples of such flows are studied.

One of the flows that is studied is the stratified Couette flow with the buoyancy frequency equal to Qy^2, where Q is a constant and y the vertical co-ordinate. The nonlinear amplitude equation is studied for various values of Q.

For the Garcia flow the nonlinear amplitude equation for the long-wave modes is evaluated, and it is compared with the corresponding equation in the Kelvin–Helmholtz flow, which has been found previously.

1. Introduction

In this paper we are concerned with the amplitude-evolution equation for linearly unstable modes in parallel shear flows of inviscid, stratified and incompressible fluids. It has been shown by Drazin (1970) and Nayfeh & Saric (1972) that the amplitude equation is second-order in time in the Kelvin–Helmholtz flow. On the other hand, however, Maslowe (1977*a*) has found that it will be first-order in time in the Holmboe flow. An obvious question is whether this difference is due to the fact that the velocity and the density profile are discontinuous in the Kelvin–Helmholtz flow, while continuous in the Holmboe flow.

In this paper it is shown that the amplitude equation may be second-order in time in continuous models as well, and two examples of such flows are studied. It depends on the dispersion relation for the linear problem whether the amplitude equation will be first- or second order in time (Benney & Maslowe 1975). In general the linear dispersion relation can be written as $\alpha^2-\alpha_s^2 = k_1(c-c_s)+k_2(c-c_s)^2+\dots$, (Engevik 1973*a*, 1975), where α_s and α are the wavenumbers and c_s and c are the wave velocities respectively of the neutral mode and the unstable mode contiguous to the neutral one; k_1 and k_2 are constants. In the two examples which we consider $k_1 = 0$, and the amplitude equations are therefore second-order in time. However, the amplitude equation will be first-order in time if the first term in the dispersion relation is the dominating term. This is in fact the case studied by Maslowe (1977*a*).

One of the flows that is studied is the stratified Couette flow with the buoyancy frequency equal to Qy^2, where Q is a constant and y the vertical co-ordinate (Høiland & Riis 1968). The nonlinear amplitude equation is studied for various values of Q.

The Garcia model (*cf.* Drazin & Howard 1966) is considered in the limiting case when $\alpha_s \to 0$ (the Kelvin–Helmholtz limit), and the amplitude equation is compared with the amplitude equation in the Kelvin–Helmholtz flow that Drazin (1970) and Nayfeh & Saric (1972) have found.

2. Derivation of the amplitude equation

We consider a parallel shear flow of a stratified, incompressible fluid with mean-velocity profile $U(y)$ and density profile $\bar{\rho}(y) = \exp(-\int^{y}\beta(y)\,dy)$; the mean velocity being in the x-direction. Both velocity and density are made dimensionless. The flow may be confined between two rigid horizontal planes at $y = y_1, y_2$, or may extend to infinity, i.e. y_1 and y_2 may become $-\infty$ and $+\infty$ respectively. Both $U(y)$ and $\beta(y)$ are assumed to be analytic functions of $y \in [y_1, y_2]$.

It is assumed that there exists a stability boundary, and the wavenumber and the wave velocity of the neutral mode ϕ_s on this stability boundary are denoted by α_s and c_s respectively. The critical layer associated with this neutral mode is at $y = y_s$, where y_s is given by the equation $U(y) = c_s$. We assume that there is only one critical layer, which lies in the interior of the flow field, and that $U'(y_s) \neq 0$, where the prime denotes differentiation with respect to y. This means that we do not consider flows with critical layers at the boundaries or the particular problems they pose (see Huppert 1973; Engevik 1978).

With the above assumption the neutral mode ϕ_s is proportional to either of the two solutions $\phi_\pm = (U-c_s)^{\frac{1}{2}\pm\nu}Y_\pm$, where $\nu = (\frac{1}{4}-J_l(y_s))^{\frac{1}{2}} \in [0, \frac{1}{2}]$. Here $J_l(y_s) = \beta(y_s)g(U'(y_s))^{-2}$ is the local Richardson number at the critical layer, $Y_\pm$ is analytic on $[y_1, y_2]$ and $Y_\pm(y_s) \neq 0$ (Miles 1961; Engevik 1973*b*). In general $\phi_\pm$ is a many-valued function, and we choose the neutral solution ϕ_s to be the branch that is given by defining $\arg(U-c_s)$ to be zero for $U-c_s > 0$ and $-\pi$ for $U-c_s < 0$ (see appendix A). When $\nu = \frac{1}{2}$, which corresponds to $J_l(y_s) = 0$, both $\phi_+ = (U-c_s)Y_+$ and $\phi_- = Y_-$ are analytic on $[y_1, y_2]$ and have no singularity at the critical layer.

The wavenumber and the wave velocity of a linearly unstable mode contiguous to the neutral one are denoted by α and c respectively. The linear dispersion relation for this mode can be written as $\alpha^2-\alpha_s^2 = k_1(c-c_s)+k_2(c-c_s)^2+\ldots$ (Engevik 1973*a*, 1975), where k_1 and k_2 are constants that are given in appendix A.

When $J_l(y_s) = 0$ the neutral solutions $\phi_+ = (U-c_s)Y_+$ and $\phi_- = Y_-$ are both analytic on $[y_1, y_2]$ as mentioned previously. This is the case for the two flows that are studied in §§3 and 4. In these two models there exist neutral solutions with $c_s = 0$; the mode that corresponds to ϕ_+ is antisymmetric, and the one corresponding to ϕ_- is symmetric. Both models belong to a class of flows where $U(y)$ is antisymmetric with respect to y, $\beta(y)$ is symmetric with $\beta(0) = 0$, and $y_1 = -y_2$. If there exist neutral modes with $c_s = 0$ for such a flow, ϕ_+ will be antisymmetric and ϕ_- symmetric. It follows from the expression for k_1 given in appendix A that $k_1 = 0$ when $\phi_s = \phi_+$, and that k_1 is purely imaginary when $\phi_s = \phi_-$. (Because of the analyticity and the symmetry properties $\beta(y) = ay^2+\ldots$ and $U(y) = b_1y+b_3y^3+\ldots$ near $y = 0$, where a, b_1 and b_3 are constants. Therefore the integrand of the integral I_1 in the expression for k_1 has no singularity at $y = 0$ when $\phi_s = \phi_+$, and it has a pole at $y = 0$ when $\phi_s = \phi_-$.)

The purpose of this paper is to investigate how a linearly unstable mode contiguous to $\phi_s = (U-c_s)Y_+$, and for which $k_1 = 0$, evolves in time. As we have shown, there exist flows for which k_1 becomes equal to zero, and two examples are studied in §§3 and 4. The linear solution of the perturbation stream function is written as $\epsilon\{\Phi_1(y,\tau)\,e^{i\alpha(x-c_S t)}+\text{c.c.}\}$, where c.c. means complex-conjugate. ϵ^2 is a small quantity that represents the order of magnitude of the perturbation kinetic energy (see Maslowe 1977*a*), and τ is a slow time scale defined by $\tau=\alpha\mu t$, where μ will subsequently be related to ϵ.

Making the Boussinesq approximation, the perturbation stream function $\epsilon\hat{\psi}$ and the density $\epsilon\hat{\rho}$ satisfy the equations

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\nabla^2\hat{\psi}-U''\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\hat{\psi}_x+\beta g\hat{\psi}_{xx}$$
$$+\epsilon\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)[\hat{\psi}_y\nabla^2\hat{\psi}_x-\hat{\psi}_x\nabla^2\hat{\psi}_y]+\epsilon\frac{g}{\bar{\rho}}\frac{\partial}{\partial x}[\hat{\psi}_y\hat{\rho}_x-\hat{\psi}_x\hat{\rho}_y]=0, \quad (2.1)$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\hat{\rho}=\bar{\rho}'\hat{\psi}_x-\epsilon(\hat{\psi}_y\hat{\rho}_x-\hat{\psi}_x\hat{\rho}_y), \quad (2.2)$$

where $\nabla^2\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$, and the subscripts x and y denote differentiation with respect to x and y respectively. At the boundaries the inviscid boundary conditions apply.

The perturbation stream function and the density are expanded as

$$\begin{aligned}\epsilon\hat{\psi}=&\,[\{\epsilon\Phi_1(y,\tau)+\epsilon^3\Phi_3(y,\tau)+\ldots\}e^{i\alpha(x-c_S t)}+\text{c.c.}]\\&+[\{\epsilon^2\Phi_2(y,\tau)+\ldots\}e^{2i\alpha(x-c_S t)}+\text{c.c.}]\\&+[\epsilon^2\Phi_{20}(y,\tau)+\ldots]+\ldots,\end{aligned} \quad (2.3)$$

$$\begin{aligned}\epsilon\hat{\rho}=&\,[\{\epsilon\rho_1(y,\tau)+\epsilon^3\rho_3(y,\tau)+\ldots\}e^{i\alpha(x-c_S t)}+\text{c.c.}]\\&+[\{\epsilon^2\rho_2(y,\tau)+\ldots\}e^{2i\alpha(x-c_S t)}+\text{c.c.}]\\&+[\epsilon^2\rho_{20}(y,\tau)+\ldots]+\ldots.\end{aligned} \quad (2.4)$$

We introduce (2.3) and (2.4) into (2.1) and obtain

$$\begin{aligned}&[\{(U-c_S)^2\nabla_1^2\Phi_1-U''(U-c_S)\,\Phi_1+\beta g\Phi_1\}-i\mu\{2(U-c_S)\,\nabla_1^2\Phi_{1\tau}-U''\Phi_{1\tau}\}-\mu^2\nabla_1^2\Phi_{1\tau\tau}\\&\quad+\epsilon^2\{(U-c_S)^2\nabla_1^2\Phi_3-U''(U-c_S)\,\Phi_3+\beta g\Phi_3+F_3(y,\tau)\}+\ldots]\,e^{i\alpha(x-c_S t)}\\&\quad+\epsilon[4\{(U-c_S)^2\nabla_2^2\Phi_2-U''(U-c_S)\,\Phi_2+\beta g\Phi_2+F_1(y,\tau)\}+\ldots]\,e^{2i\alpha(x-c_S t)}\\&\qquad\qquad-\epsilon\mu[\{\mu\Phi_{20yy\tau}+F_2(y,\tau)\}_\tau+\ldots]+\ldots=0,\end{aligned} \quad (2.5)$$

where

$$\nabla_1^2\equiv\frac{d^2}{dy^2}-\alpha^2,\quad \nabla_2^2\equiv\frac{d^2}{dy^2}-4\alpha^2,$$

and the subscript τ denotes differentiation with respect to τ.

$$\left.\begin{aligned}F_1(y,\tau)&=\tfrac{1}{2}(U-c_S)\,[\Phi_{1y}\nabla_1^2\Phi_1-\Phi_1\nabla_1^2\Phi_{1y}]+\tfrac{1}{2}\frac{g}{\bar{\rho}}[\Phi_{1y}\rho_1-\Phi_1\rho_{1y}],\\F_2(y,\tau)&=-i[\Phi_{1y}\nabla_1^2\Phi_1^*-\Phi_{1y}^*\nabla_1^2\Phi_1+\Phi_1\nabla_1^2\Phi_{1y}^*-\Phi_1^*\nabla_1^2\Phi_{1y}],\\F_3(y,\tau)&=F_{31}(y,\tau)+F_{32}(y,\tau),\end{aligned}\right\} \quad (2.6a)$$

where

$$\left.\begin{aligned} F_{31}(y,\tau) &= (U-c_s)\,[2\Phi^*_{1y}\nabla^2_2\Phi_2 - 2\Phi_2\nabla^2_1\Phi^*_{1y} + \Phi^*_1\nabla^2_2\Phi_{2y} - \Phi_{2y}\nabla^2_1\Phi^*_1] \\ &\qquad + \frac{g}{\bar\rho}[2\Phi^*_{1y}\rho_2 - 2\Phi_2\rho^*_{1y} + \Phi^*_1\rho_{2y} - \Phi_{2y}\rho^*_1], \\ F_{32}(y,\tau) &= (U-c_s)\,[\Phi_{20y}\nabla^2_1\Phi_1 - \Phi_1\Phi_{20yyy}] + \frac{g}{\bar\rho}[\Phi_{20y}\rho_1 - \Phi_1\rho_{20y}], \end{aligned}\right\} \tag{2.6b}$$

where the asterisk means complex conjugate.

The equations for ρ_1, ρ_2 and ρ_{20} are obtained from (2.2), i.e.

$$(U-c_s)\,\rho_1 = \bar\rho'\Phi_1, \tag{2.7}$$

$$(U-c_s)\,\rho_2 = \bar\rho'\Phi_2 - \tfrac{1}{2}[\Phi_{1y}\rho_1 - \Phi_1\rho_{1y}], \tag{2.8}$$

$$\mu\rho_{20\tau} = i[\Phi_{1y}\rho^*_1 - \Phi^*_{1y}\rho_1 + \Phi_1\rho^*_{1y} - \Phi^*_1\rho_{1y}]. \tag{2.9}$$

The quantities F_{31} and F_{32} represent the nonlinear interactions of the fundamental mode with the second harmonic and with the mean-flow distortion respectively. The linear solution is expanded in powers of μ, and we write

$$\left.\begin{aligned} \Phi_1(y,\tau) &= A(\tau)\,\phi_s(y) + \mu\frac{dA}{d\tau}\phi_{12}(y) + \mu^2\frac{d^2A}{d\tau^2}\phi_{13}(y) + \dots, \\ \rho_1(y,\tau) &= A(\tau)\,\rho_s(y) + \mu\frac{dA}{d\tau}\rho_{12}(y) + \mu^2\frac{d^2A}{d\tau^2}\rho_{13}(y) + \dots, \\ \Phi_2(y,\tau) &= A^2\phi_{21}(y), \quad \rho_2(y,\tau) = A^2\rho_{21}(y), \quad \Phi_{20}(y,\tau) = AA^*\phi_{201}(y), \\ \rho_{20}(y,\tau) &= AA^*\rho_{201}(y), \quad \Phi_3(y,\tau) = A^2A^*\phi_{31}(y), \quad \rho_3(y,\tau) = A^2A^*\rho_{31}(y). \end{aligned}\right\} \tag{2.10}$$

If (2.5) is to be satisfied, the coefficients of $\exp\{i\alpha(x-c_st)\}$, $\exp\{2i\alpha(x-c_st)\}$ and $\exp\{i0\}$ must all be equal to zero. We introduce the expressions given in (2.10) into the coefficients of $\exp\{i\alpha(x-c_st)\}$ and $\exp\{2i\alpha(x-c_st)\}$, and get the equations

$$\begin{aligned} &AL_1\phi_s - (\alpha^2-\alpha_s^2)\,\phi_s A + \mu\left[L_1\phi_{12} - i\left(\frac{2}{U-c_s}\nabla^2_{10}\phi_s - \frac{U''}{(U-c_s)^2}\phi_s\right)\right]\frac{dA}{d\tau} \\ &\quad + \mu^2\left[L_1\phi_{13} - i\left(\frac{2}{U-c_s}\nabla^2_{10}\phi_{12} - \frac{U''}{(U-c_s)^2}\phi_{12}\right) - \frac{1}{(U-c_s)^2}\nabla^2_{10}\phi_s\right]\frac{d^2A}{d\tau^2} \\ &\quad - (\alpha^2-\alpha_s^2)\,\mu\left[\phi_{12} - i\frac{2\phi_s}{U-c_s}\right]\frac{dA}{d\tau} + \epsilon^2A^2A^*[L_1\phi_{31} - H(y)] + \dots = 0, \end{aligned} \tag{2.11}$$

$$L_2\phi_{21} = G(y), \tag{2.12}$$

where

$$\nabla^2_{10} \equiv \frac{d^2}{dy^2} - \alpha_s^2, \quad \nabla^2_{20} \equiv \frac{d^2}{dy^2} - 4\alpha_s^2,$$

$$L_1 \equiv \nabla^2_{10} + \frac{\beta g}{(U-c_s)^2} - \frac{U''}{U-c_s}, \quad L_2 \equiv \nabla^2_{20} + \frac{\beta g}{(U-c_s)^2} - \frac{U''}{U-c_s}.$$

The mean-flow distortion must satisfy the equation (see appendix B)

$$\mu\, d/d\tau(AA^*)\,\phi''_{201} = -F_2(y,\tau), \tag{2.13}$$

which gives the coefficient of zero for $\exp\{i0\}$ in (2.5).

The quantities $G(y)$ and $H(y)$ in (2.11) and (2.12) are the nonlinear terms that are obtained by introducing the expressions (2.10) into the expressions for F_1 and F_3 in (2.6):

$$G(y) = -\frac{1}{2(U-c_s)}[\phi_s'\nabla_{10}^2\phi_s - \phi_s\nabla_{10}^2\phi_s'] - \frac{g}{2\bar{\rho}(U-c_s)^2}[\phi_s'\rho_s - \phi_s\rho_s'], \tag{2.14}$$

$$H(y) = H_1(y) + H_2(y),$$

where

$$\left.\begin{aligned} H_1(y) &= -\frac{1}{U-c_s}[2\phi_s^{*\prime}\nabla_{20}^2\phi_{21} - 2\phi_{21}\nabla_{10}^2\phi_s^{*\prime} + \phi_s^*\nabla_{20}^2\phi_{21}' - \phi_{21}'\nabla_{10}^2\phi_s^*] \\ &\quad - \frac{g}{\bar{\rho}(U-c_s)^2}[2\phi_s^{*\prime}\rho_{21} - 2\phi_{21}\rho_s^{*\prime} + \phi_s^*\rho_{21}' - \phi_{21}'\rho_s^*], \\ H_2(y) &= -\frac{1}{U-c_s}[\phi_{201}'\nabla_{10}^2\phi_s - \phi_s\phi_{201}'''] - \frac{g}{\bar{\rho}(U-c_s)^2}[\phi_{201}'\rho_s - \phi_s\rho_{201}']. \end{aligned}\right\} \tag{2.15}$$

ϕ_s satisfies the equation

$$L_1\phi_s = 0, \tag{2.16}$$

which follows from (2.11), and ρ_s and ρ_{21} are given by the equations

$$(U-c_s)\rho_s = \bar{\rho}'\phi_s, \tag{2.17}$$

$$(U-c_s)\rho_{21} = \bar{\rho}'\phi_{21} - \tfrac{1}{2}[\phi_s'\rho_s - \phi_s\rho_s'], \tag{2.18}$$

which are found by introducing the expressions (2.10) into (2.7) and (2.8).

In appendix B it is shown that the equation for ϕ_{201} is

$$\phi_{201}'' = \left[\left(-\frac{2\beta g}{(U-c_s)^3} + \frac{U''}{(U-c_s)^2}\right)\phi_s^2\right]', \tag{2.19}$$

which follows from (2.13). It is also found that

$$\rho_{201} = 0. \tag{2.20}$$

If we use (2.16) in (2.11) we get

$$\begin{aligned} &-(\alpha^2-\alpha_s^2)\phi_s A + \mu\left\{L_1\phi_{12} + i\left(\frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2}\right)\phi_s\right\}\frac{dA}{d\tau} \\ &\quad + \mu^2\left\{L_1\phi_{13} - i\left(\frac{2}{U-c_s}\nabla_{10}^2\phi_{12} - \frac{U''}{(U-c_s)^2}\phi_{12}\right) - \frac{1}{(U-c_s)^2}\nabla_{10}^2\phi_s\right\}\frac{d^2A}{d\tau^2} \\ &\quad - (\alpha^2-\alpha_s^2)\mu\left\{\phi_{12} - i\frac{2\phi_s}{U-c_s}\right\}\frac{dA}{d\tau} + \epsilon^2A^2A^*\{L_1\phi_{31} - H(y)\} + \ldots = 0. \end{aligned} \tag{2.21}$$

The terms multiplied by μ and μ^2 are connected with the linear solution of the problem. From (2.21) we can derive the dispersion relation for the linear problem in the general case by neglecting the nonlinear terms. Let us show this before we proceed with the nonlinear theory for the cases we are interested in. We multiply (2.21) by ϕ_s and subtract from it the two expressions that are obtained by multiplying (2.16) by $\mu\phi_{12}$ and by $\mu^2\phi_{13}$. The result is integrated from y_1 to y_2 along the contour L which goes around the critical point y_s in the correct way (see appendix A). If, in the linear case, we define μ by $i\mu = c - c_s$, then A is proportional to e^{τ}, and we obtain the following expression for the dispersion relation in the general case

$$\alpha^2 - \alpha_s^2 = i\mu k_1 - \mu^2 k_2 + \ldots, \tag{2.22}$$

where the expression for k_1 is the same as that in appendix A, and

$$k_2 = \left[\int_L \left\{ i\left(\frac{2}{U-c_s}\nabla_{10}^2\phi_{12} - \frac{U''}{(U-c_s)^2}\phi_{12} + k_1\phi_{12}\right) + \left(\frac{1}{(U-c_s)^2}\nabla_{10}^2\phi_s + \frac{2k_1\phi_s}{U-c_s}\right)\right\}\phi_s\, dy\right]\left\{\int_L \phi_s^2\, dy\right\}^{-1}. \quad (2.23)$$

We see that in the linear case we have defined μ in such a way that it may be complex. However, for many of the flows that have been studied, it is found that $c - c_s$ is purely imaginary, and then μ is real. This applies to the two examples which are studied in §§3 and 4, and also for many of the examples given in Drazin & Howard (1966).

The equations for ϕ_{12} and ϕ_{13} are found by introducing the expression (2.22) for $\alpha^2 - \alpha_s^2$ into the linear version of (2.21) with A proportional to e^τ. If the coefficients of μ and μ^2 are to be zero, ϕ_{12} and ϕ_{13} must satisfy the equations

$$L_1\phi_{12} = -i\left(\frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2} - k_1\right)\phi_s = iJ_1, \quad (2.24)$$

$$L_1\phi_{13} = i\left(\frac{2}{U-c_s}\nabla_{10}^2\phi_{12} - \frac{U''}{(U-c_s)^2}\phi_{12} + k_1\phi_{12}\right) + \left(\frac{1}{(U-c_s)^2}\nabla_{10}^2\phi_s + \frac{2k_1\phi_s}{U-c_s}\right) - k_2\phi_s. \quad (2.25)$$

Both equations have solutions, since the solution to the corresponding adjoint problem is orthogonal to the right-hand side of the equations. The differential operator L_1 is self-adjoint, so that the adjoint function that satisfies the proper boundary conditions is ϕ_s. (This is equivalent to multiplying (2.24) or (2.25) by ϕ_s and (2.16) by ϕ_{12} or ϕ_{13}, subtracting the one expression from the other and integrating along the contour L.) The solution of (2.24) that satisfies the boundary conditions is

$$\phi_{12} = C\phi_s + i\left[\phi_s\int_{y_1}^{y}\frac{J_1\theta_s}{W}dt + \theta_s\int_{y}^{y_2}\frac{J_1\phi_s}{W}dt\right], \quad (2.26)$$

where C is a constant, ϕ_s and θ_s are two linearly independent solutions of (2.16), W is the Wronskian (which is a constant), and the integration is along L. We put $C = 0$ because $C \neq 0$ will give rise to a term which may be included in the term $A\phi_s$ in (2.10), and means only a redefinition of that term. We write

$$\phi_{12} = i\theta_1, \quad \text{where} \quad \theta_1 = \phi_s\int_{y_1}^{y}\frac{J_1\theta_s}{W}dt + \theta_s\int_{y}^{y_2}\frac{J_1\phi_s}{W}dt. \quad (2.27)$$

k_2 given by (2.23) can be transformed into the expression given in appendix A by using (2.24) and (2.27). The relation (2.22) is therefore equal to the relation (A 2) in appendix A because $i\mu = c - c_s$.

After this digression let us go back to the problem that we stated at the beginning of this section. We put $k_1 = 0$ in (2.24) and (2.25), and introduce ϕ_{12} and ϕ_{13} given by these equations into (2.21) to obtain

$$-(\alpha^2-\alpha_s^2)\phi_s A - \mu^2 k_2\phi_s\frac{d^2A}{d\tau^2} - (\alpha^2-\alpha_s^2)\mu\left(\phi_{12} - \frac{2i\phi_s}{U-c_s}\right)\frac{dA}{d\tau} + \epsilon^2 A^2 A^*(L_1\phi_{31} - H(y)) + \ldots = 0. \quad (2.28)$$

We see that the integrands of the integrals in the expressions for k_1 and k_2 have no singularities on $[y_1, y_2]$ if $\beta = a(y-y_s)^2 + \ldots$ and $U = b_1(y-y_s) + b_3(y-y_s)^3 + \ldots$ in the vicinity of the critical layer y_s, where a, b_1 and b_3 are constants. We also see from (2.12) and (2.19) that ϕ_{21} and ϕ_{201} have no singularities on $[y_1, y_2]$. This applies to the two flows that we are studying in §§3 and 4, and for other flows having the same type of velocity and density profiles (see the discussion at the beginning of this section).

Now we multiply (2.28) by ϕ_s and (2.16) by ϕ_{31}, subtract the one expression from the other, and integrate along the real axis from y_1 to y_2. We obtain

$$-(\alpha^2-\alpha_s^2)\,A - \mu^2 k_2 \frac{d^2A}{d\tau^2} - \epsilon^2 C A^2 A^* + O((\alpha^2-\alpha_s^2)\,\mu, \epsilon^4) = 0, \tag{2.29}$$

where

$$C = C_1 + C_2, \quad C_i = \int_{y_1}^{y_2} H_i(y)\,\phi_s\,dy \bigg/ \int_{y_1}^{y_2} \phi_s^2\,dy \quad (i = 1, 2). \tag{2.30}$$

We find that the integrals in the numerators in the expressions for C_1 and C_2 exist.

Assume that $\mu^2 = O(\alpha^2 - \alpha_s^2) = O(\epsilon^2)$, which means that the slow time scale $\tau = \alpha\epsilon t$. It follows from (2.29) that

$$k_2 \frac{d^2A}{dt^2} + \alpha_s^2(\alpha^2-\alpha_s^2)\,A + \epsilon^2\alpha_s^2 C A^2 A^* = 0, \tag{2.31}$$

where we have written the amplitude equation in terms of the original fast time scale. This amplitude equation is similar to the equation found by Drazin (1970) and Nayfeh & Saric (1972) in the Kelvin–Helmholtz flow. Recently Weissman (1979) has given a brief review of the various solutions of (2.31) allowing for arbitrary values of the constants in the equation.

Maslowe (1977*a*) obtained an amplitude equation of first order in time in the Holmboe flow (cf. Drazin & Howard 1966); the amplitude evolving on the slow time scale $\tau = \alpha\epsilon^2 t$. This is because the first term in the dispersion relation for this flow is the dominating term for almost all α_s. However, it can be shown by using the formulae in appendix A that for the Holmboe flow $k_1 \sim -2\pi i \alpha_s^3$ and $k_2 \sim -2\alpha_s^2$ when $\alpha_s \to 0$ (i.e. the Kelvin–Helmholtz limit). It means that k_1 tends to zero with α_s faster than k_2 does, which is to be expected from the results of Drazin & Howard (1961, 1963) concerning the stability characteristics of unbounded flows for long waves (cf. Drazin & Howard 1966). Therefore, when α_s becomes small enough it is to be expected that the second term in the dispersion relation will be the dominating term, and then the amplitude equation will be second-order in time.

3. Example I

We consider the case $U = y$, $\beta g = R_0 + Qy^2$ and $y_2 = -y_1 = 1$, where $R_0 \geqslant 0$ and $Q \geqslant 0$ (Høiland & Riis 1968). Let J_μ denote the Bessel function of order μ, $\lambda_{j,\mu}$ the jth zero of J_μ, and $\nu = (\frac{1}{4} - R_0)^{\frac{1}{2}}$. When $0 \leqslant R_0 \leqslant \frac{1}{4}$ there exist the neutral modes (Engevik 1973*b*)

$$\left.\begin{aligned} c_s = 0, \quad \phi_s &= y^{\frac{1}{2}} J_\nu(\lambda_{j,\nu} y), \quad \alpha_s^2 = \alpha_{j,\nu}^2 = Q - \lambda_{j,\nu}^2 \quad (j = 1, 2, \ldots, n_1);\\ c_s = 0, \quad \phi_s &= y^{\frac{1}{2}} J_{-\nu}(\lambda_{j,-\nu} y), \quad \alpha_s^2 = \alpha_{j,-\nu}^2 = Q - \lambda_{j,-\nu}^2 \quad (j = 1, 2, \ldots, n_2);\end{aligned}\right\} \tag{3.1}$$

where n_1 and n_2 are the largest integers that satisfy $Q-\lambda^2_{n_1,\nu} \geqslant 0$ and $Q-\lambda^2_{n_2,-\nu} \geqslant 0$ respectively. For $R_0 = 0$ the solutions in (3.1) can be written as

$$\phi_s = \sin j\pi y, \quad \alpha_s^2 = \alpha_{j,\frac{1}{2}}^2 = Q-(j\pi)^2, \tag{3.2}$$

corresponding to the + sign in (3.1), and

$$\phi_s = \cos(j-\tfrac{1}{2})\pi, \quad \alpha_s^2 = \alpha_{j,-\frac{1}{2}}^2 = Q-((j-\tfrac{1}{2})\pi)^2, \tag{3.3}$$

corresponding to the − sign. The modes in (3.2) and (3.3) are also given by Høiland & Riis (1968). In Engevik (1973*b*) is found that for various values of $Q, k_1 \neq 0$ for all the modes in (3.1) except for those given in (3.2). For the modes in (3.2) $k_1 = 0$, and k_2, the general formula for which is given in appendix A, is found to be

$$k_2 = 6Qj\pi\,\mathrm{Si}\,(2\pi j) - \frac{2Q^2}{j\pi}\,\mathrm{Si}\,(2\pi j)\,\mathrm{Cin}\,(2\pi j)$$
$$-\frac{2Q^2}{j\pi}\left[\int_0^1 \frac{1-\cos 2\pi jy}{y}\,\mathrm{Si}\,(2\pi jy)\,dy - \int_0^1 \frac{\sin 2\pi jy}{y}\,\mathrm{Cin}\,(2\pi jy)\,dy\right], \tag{3.4}$$

where the sine and cosine integrals are defined respectively by

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t}\,dt, \quad \mathrm{Cin}\,(x) = \int_0^x \frac{1-\cos t}{t}\,dt.$$

Huppert (1973) found that $(\partial c/\partial \alpha)_Q = \infty$ for the modes in (3.2) by using Howard's (1963) formula. This is in agreement with the result $k_1 = 0$ of Engevik (1973*b*), because the formula for k_1 given in appendix A is the inverse of Howard's formula. Huppert in fact also calculated the inverse of k_2 and found an expression which has been shown to be equivalent to that of (3.4) (Banks & Drazin 1973; Engevik 1973*a*).

ϕ_{21} has to satisfy (2.12), and when the neutral mode is given by (3.2) we get

$$\phi_{21}'' + m^2\phi_{21} = \frac{1}{2}\left(\frac{Q^2 y}{g} - \frac{Q}{y^2}\right)\sin^2 j\pi y, \tag{3.5}$$

where $m^2 = Q - 4\alpha_s^2$. The boundary conditions are

$$\phi_{21} = 0 \quad \text{at} \quad y = \pm 1. \tag{3.6}$$

We see that m^2 can be positive, negative or equal to zero. A solution to (3.5) subjected to the boundary conditions (3.6) is not obtainable when both α_s and $2\alpha_s$ are eigenvalues of (2.16). When a solution exists, it is easily found, so we do not write down the expression for ϕ_{21}.

The mean-flow distortion ϕ_{201}' is found from (2.19), i.e.

$$\phi_{201}' = -(2Q/y)\sin^2 j\pi y, \tag{3.7}$$

where we have put the constant of integration equal to zero. A non-zero constant of integration will give no contribution to the constant C_2 in (2.30).

The amplitude equation becomes

$$d^2A/dt^2 = a_0\alpha_s^2(\alpha^2-\alpha_s^2)A + \epsilon^2\alpha_s^2 a_2 A^2 A^*, \tag{3.8}$$

where $a_0 = -k_2^{-1}$, $a_2 = a_{21} + a_{22}$ and $a_{21} = -C_1/k_2$, $a_{22} = -C_2/k_2$. a_0 and a_2 are real constants.

The coefficient of A in (3.8) is positive. The equation is linearly unstable, but when

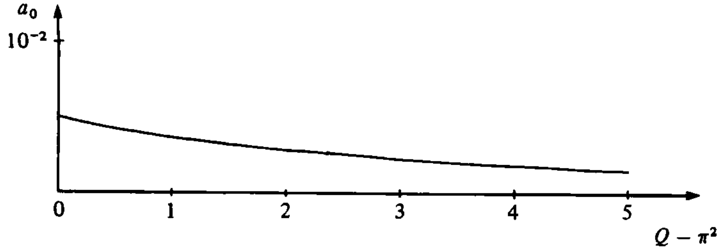

FIGURE 1. The constant a_0 as a function of Q.

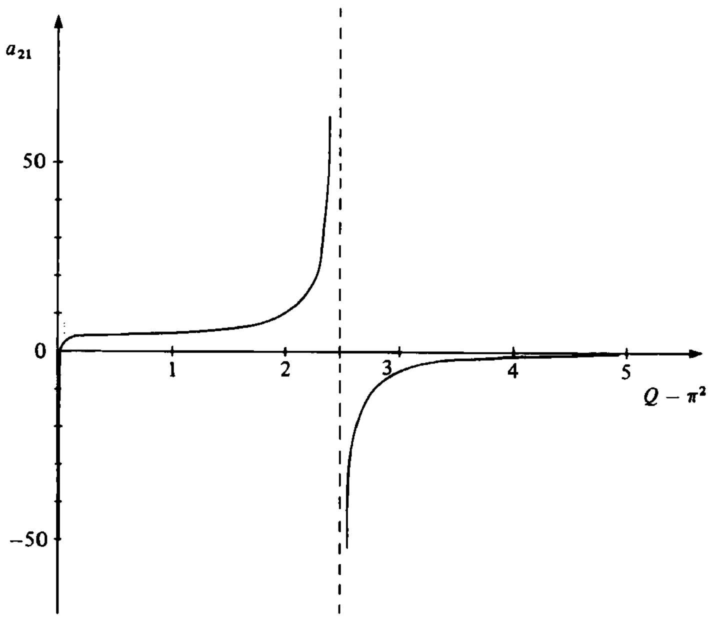

FIGURE 2. The constant a_{21} as a function of Q.

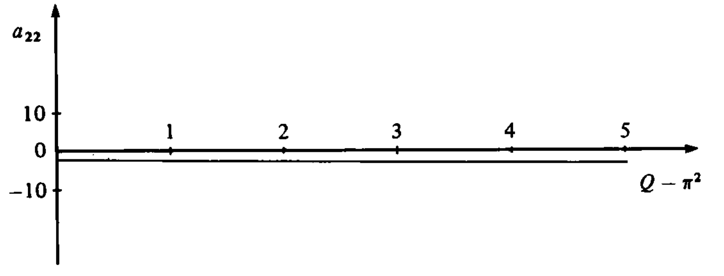

FIGURE 3. The constant a_{22} as a function of Q.

$a_2 < 0$ the nonlinear term is stabilizing; when $a_2 > 0$ the nonlinear term is destabilizing (see Weissman 1979).

We have made some numerical calculations of k_2, a_0, a_{21} and a_{22}, associated with the first mode in (3.2), for various values of Q. The results are shown in figures 1–4. It is found that k_2 is negative, so that there are linearly unstable modes for $\alpha > \alpha_{1,\frac{1}{2}} = (Q-\pi^2)^{\frac{1}{2}}$.

a_{21} represents the interaction of the fundamental mode with the second harmonic. We see from figure 2 that a_{21} is singular for $Q = \pi^2$ and $Q = \frac{5}{4}\pi^2$, which correspond to

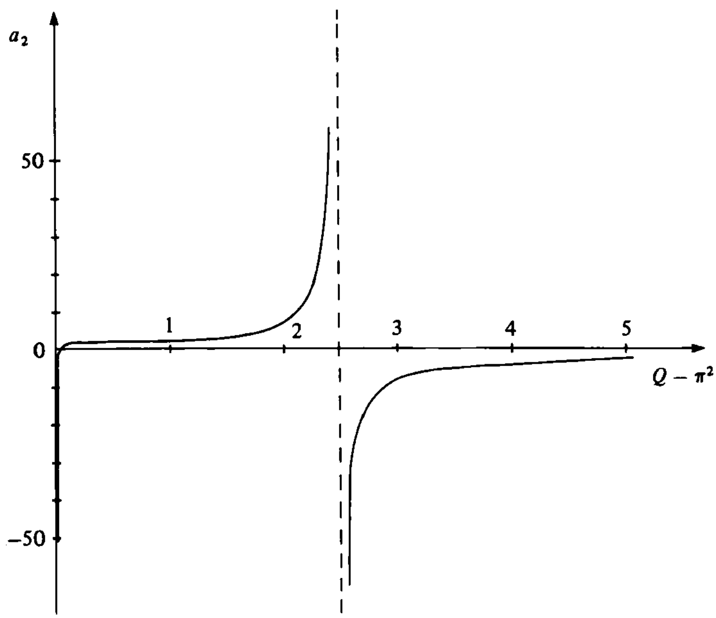

FIGURE 4. The constant a_2 as a function of Q.

the wavenumbers $\alpha_s = 0$ and $\alpha_s = \frac{1}{2}\pi$ respectively. The breakdown of the theory is due to a resonance which occurs because both α_s and $2\alpha_s$ are eigenvalues of (2.16) for these particular values of Q, as mentioned previously. The right-hand side of (2.12) is generally non-zero and solutions of (2.12) are therefore not obtainable for these particular values of Q. Maslowe (1977a) also found a case of resonance in the Holmboe flow.

a_{22} represents the effect of the mean-flow distortion (figure 3). It is negative and is therefore stabilizing. Figure 4 shows $a_2 = a_{21} + a_{22}$.

The calculations were done on a UNIVAC 1110. In calculating definite integrals we have used a NAG FORTRAN routine which evaluates the integrals using Romberg's method. NAG FORTRAN routines for calculating Si (x) and Cin (x) have also been used.

4. Example II

In this example we will consider the model $U = \tanh y$, $\beta g = 3J_0 \operatorname{sech}^2 y \tanh^2 y$ and $y_2 = -y_1 = \infty$, which was first studied by Garcia (cf. Holmboe 1962; Miles 1963). Garcia found the neutral modes

$$c_s = 0, \quad J_0 = \tfrac{1}{3}\alpha_s(\alpha_s + 3), \quad \phi_s = \tanh y\,(\operatorname{sech} y)^{\alpha_s}, \tag{4.1}$$

$$c_s = 0, \quad J_0 = \tfrac{1}{3}(\alpha_s - 1)(\alpha_s + 2), \quad \phi_s = (\operatorname{sech} y)^{\alpha_s}, \tag{4.2}$$

which define a stability boundary.

Miles (1963) found that there exists an infinite number of distinct branches of the stability boundary for this model.

For the neutral mode in (4.1) $k_1 = 0$. In appendix A k_2 has been written as $k_2 = (I_2 + I_3)/I_0$, where I_2 can be expressed as

$$I_2 = \int_L X\phi_s^2 dy \int_{-\infty}^{y} \phi_s^{-2} dt \int_t^{\infty} X\phi_s^2 du, \tag{4.3}$$

where

$$X = \frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2} - k_1,$$

and the integrations are along the contour L defined in appendix A.

Introducing ϕ_s given by (4.1) into the expressions for I_0 and I_3 given in appendix A, and I_2 given by (4.3), we obtain

$$\left.\begin{aligned} I_0 &= B(\tfrac{3}{2}, \alpha_s), \\ I_2 &= -2(1+3\alpha_s+\alpha_s^2)^2 B(\tfrac{1}{2}, \alpha_s+1)/(1+\alpha_s), \\ I_3 &= (2+9\alpha_s+3\alpha_s^2)\, B(\tfrac{1}{2}, \alpha_s+1), \end{aligned}\right\} \tag{4.4}$$

where $B(r, s)$ denotes the beta function. Using the expressions in (4.4) we find that

$$k_2 \sim -2\alpha_s^2 \quad \text{when} \quad \alpha_s \to 0, \tag{4.5}$$

and the dispersion relation in appendix A yields

$$c^2 \sim -(\alpha^2-\alpha_s^2)/2\alpha_s^2 \sim -(\alpha-\alpha_s)/\alpha_s \quad \text{when} \quad \alpha-\alpha_s \ll \alpha_s, \quad \alpha_s \to 0. \tag{4.6}$$

Equation (4.6) yields instability on the side of the stability boundary for which $\alpha > \alpha_s$. If we put $\alpha - \alpha_s \ll \alpha_s$ into the dispersion relation for the Kelvin–Helmholtz flow, we obtain the expression given in (4.6). Therefore, as far as the linear theory is concerned, the Kelvin–Helmholtz flow is a good model for the smoothly varying Garcia flow for modes with wavenumbers satisfying the condition $\alpha - \alpha_s \ll \alpha_s$. We also notice that the expression for k_2 given in (4.5) is the same as that found for the Holmboe flow (cf. the discussion at the end of §2).

Before we proceed with the nonlinear theory we will calculate k_1 associated with the neutral mode in (4.2). By using the formulae in appendix A, we find that

$$\left.\begin{aligned} k_1 &= i(2+6J_0)\pi/B(\tfrac{1}{2}, \alpha_s), \\ c &= -iB(\tfrac{1}{2}, \alpha_s)\,(\alpha^2-\alpha_s^2)/(2+6J_0)\pi + \ldots, \end{aligned}\right\} \tag{4.7}$$

which yields instability on the side of the stability boundary for which $\alpha < \alpha_s$. Equations (4.6) and (4.7) of course predict instability for the same wavenumbers as given by Garcia (cf. Drazin & Howard 1966).

Let us now find the amplitude equation for a linearly unstable mode contiguous to the neutral one in (4.1) when α_s is small. The equation for ϕ_{21} becomes

$$\left[\frac{d^2}{dy^2} - 4\alpha_s^2 + (2+3\alpha_s+\alpha_s^2)\operatorname{sech}^2 y\right]\phi_{21} = (2+6\alpha_s+2\alpha_s^2)(\operatorname{sech} y)^{2+2\alpha_s} - (2+\tfrac{15}{2}\alpha_s+\tfrac{5}{2}\alpha_s^2)(\operatorname{sech} y)^{4+2\alpha_s}, \tag{4.8}$$

with the boundary conditions

$$\phi_{21} \to 0 \quad \text{when} \quad y \to \pm\infty. \tag{4.9}$$

In (4.8) we have neglected the effect of the variation of the inertia due to the heterogeneity of the fluid as being small compared to the effect of the buoyancy.

The solution of (4.8) that satisfies the boundary conditions (4.9), is

$$\phi_{21} = (\tfrac{1}{2}+\alpha_s-\tfrac{3}{2}\alpha_s^2+O(\alpha_s^3))(\operatorname{sech} y)^{2+2\alpha_s}-(\alpha_s-\tfrac{1}{2}\alpha_s^2+O(\alpha_s^3))(\operatorname{sech} y)^{2\alpha_s} \quad \text{when} \quad \alpha_s \to 0. \tag{4.10}$$

The mean-flow distortion ϕ'_{201} is given by (2.19), and becomes

$$\phi'_{201} = -(2+6\alpha_s+2\alpha_s^2)\tanh y\,(\operatorname{sech} y)^{2+2\alpha_s}, \tag{4.11}$$

where we have used the fact that $\phi'_{201} \to 0$ when $y \to \pm\infty$.

We introduce ϕ_s, ϕ_{21} and ϕ'_{201} given by (4.1), (4.10) and (4.11) into the integrals in the numerators of the expressions for C_1 and C_2 in (2.30). The denominators are equal to I_0 given by (4.4). After some calculations we obtain

$$C_1 \sim \tfrac{8}{3}\alpha_s^3, \quad C_2 \sim -\tfrac{64}{15}\alpha_s^3 \quad \text{when} \quad \alpha_s \to 0, \tag{4.12}$$

and the amplitude equation becomes

$$d^2A/dt^2 = \alpha_s(\alpha-\alpha_s)A-\epsilon^2(\tfrac{4}{5}\alpha_s^3)A^2A^*. \tag{4.13}$$

We see from (4.12) that while the effect of the interaction of the fundamental mode with the second harmonic is destabilizing, the effect of the interaction of the fundamental mode with the mean-flow distortion is stabilizing.

The amplitude equation (4.13) is similar to that found by Drazin (1970) and Nayfeh & Saric (1972) in the Kelvin–Helmholtz flow. The linear part of our equation is found to be in agreement with the linear parts of the equations of Drazin and Nayfeh & Saric when these equations are considered in the limit corresponding to the limit $\alpha_s \to 0$. However, the numerical value of the nonlinear term is not the same as in the Kelvin–Helmholtz flow. The nonlinear term is, however, of the same order of magnitude and is stabilizing, as in that case.

5. Conclusion

There exist flows with continuous velocity and density profiles where there are near-neutral, linearly unstable modes with nonlinear amplitude equations of second order in time. The Høiland & Riis model and the Garcia model are two examples of such flows. The amplitude equations are second-order in time because the linear dispersion relations for the unstable modes are given by $\alpha^2-\alpha_s^2 = k_2^2(c-c_s)^2+\ldots$ in these cases. In general it depends on the linear dispersion relation whether the amplitude equation will be first- or second-order in time. If the first term in the dispersion relation given in appendix A is the dominating term, the amplitude equation will be first-order in time, which is the case considered by Maslowe (1977*a*) who studied the Holmboe flow.

The nonlinear amplitude equation for a linearly unstable mode contiguous to the neutral one, $\sin \pi y$, in the Høiland & Riis flow is studied for various values of Q which represent an overall Richardson number. Some numerical results are presented in figures 1–4. We find that there is a region on the Q-axis where the nonlinear term is positive, and it is therefore destabilizing for these values of Q. However, the nonlinear term is stabilizing when Q becomes large enough.

The Garcia flow has been studied in the limit when $\alpha_s \to 0$ (i.e. the Kelvin–Helmholtz limit), and it is found that the linear part of the amplitude equation coincides with the linear parts of the equations obtained by Drazin (1970) and Nayfeh & Saric (1972) in the Kelvin–Helmholtz flow, when their equations are considered in the limit $\alpha_s \to 0$. The nonlinear terms in the Garcia flow and in the Kelvin–Helmholtz flow do not have the same numerical value. They are, however, of the same order of magnitude, and are stabilizing in both cases.

In this paper the diffusive effects have not been taken into account except when the contour L is chosen. A little viscosity will change the trajectories in the phase plane for the solutions of the amplitude equation (2.31), as was also pointed out by Drazin (1970) in connection with the solution he obtained in the Kelvin–Helmholtz flow. In our model this is seen by adding the diffusive terms to (2.1) and (2.2). It will give rise to an additional term proportional to dA/dt (a damping term) in the amplitude equation and this term will change the trajectories in the phase plane.

Appendix A

It is assumed that $U(y)$ and $\beta(y)$ are analytic functions on $[y_1, y_2]$, and that there is only one critical layer in the interior of the flow field, i.e. we do not consider cases with critical layers at the boundaries. The critical layer is at $y = y_s$, where $y_1 < y_s < y_2$, and we assume that $U'(y_s) \neq 0$.

The neutral mode ϕ_s, with the wave velocity c_s and the wavenumber α_s, satisfies the Taylor–Goldstein equation

$$\phi'' + \left(\frac{\beta g}{(U-c_s)^2} - \frac{U''}{U-c_s} - \alpha_s^2\right)\phi = 0. \tag{A 1}$$

Equation (A 1) has a regular singularity at the critical layer y_s. As is well-known this singularity can be removed by introducing dissipative effects. A small viscosity within the critical layer will give rise to a phase change across the layer. We consider the solutions on a contour L that goes around the critical point in the correct way, i.e. in accordance with the phase change across the layer. $\arg(U-c_s)$ is defined to be zero for $U-c_s > 0$ and $-\pi$ for $U-c_s < 0$.

With the above assumption the neutral solution ϕ_s is proportional to either of the two solutions $\phi_\pm = (U-c_s)^{\frac{1}{2}\pm\nu} Y_\pm$, where $\nu = (\frac{1}{4} - J_l(y_s))^{\frac{1}{2}} \in [0, \frac{1}{2}]$. Here $J_l(y_s) = \beta(y_s) g (U'(y_s))^{-2}$ is the local Richardson number at the critical layer $y = y_s$, $Y_\pm$ is analytic on $[y_1, y_2]$, and $Y_\pm(y_s) \neq 0$ (Miles 1961; Engevik 1973*b*). In general $\phi_\pm$ is a many-valued function, and we choose the branch for ϕ_s that is in accordance with the definition of $\arg(U-c_s)$ above, i.e. $\phi_\pm = (U-c_s)^{\frac{1}{2}\pm\nu} Y_\pm$ for $U-c_s > 0$ and $\phi_\pm = \exp\{-i\pi(\frac{1}{2}\pm\nu)\}|U-c_s|^{\frac{1}{2}\pm\nu} Y_\pm$ for $U-c_s < 0$. $\phi_\pm$ is analytic on L.

In Engevik (1973*a*, 1975) the linear dispersion relation for an unstable mode contiguous to the neutral one is written as

$$\alpha^2 - \alpha_s^2 = k_1(c-c_s) + k_2(c-c_s)^2 + \dots, \tag{A 2}$$

where

$$k_1 = I_1/I_0, \quad k_2 = (I_2 + I_3)/I_0, \tag{A 3}$$

$$I_0 = \int_L \phi_s^2\, dy, \quad I_1 = \int_L \left(\frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2}\right)\phi_s^2\, dy,$$

$$I_2 = \int_L \left(\frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2} - k_1\right)\theta_1 \phi_s\, dy,$$

$$I_3 = \int_L \left(\frac{3\beta g}{(U-c_s)^4} - \frac{U''}{(U-c_s)^3}\right)\phi_s^2\,dy,$$

$$\theta_1 = \phi_s \int_{y_1}^{y} \frac{J_1\theta_s}{W}\,dt + \theta_s \int_{y}^{y_1} \frac{J_1\phi_s}{W}\,dt,$$

where the integration is along L. θ_s is a solution of (A 1), and θ_s and ϕ_s are linearly independent solutions. W is the Wronskian, which is a constant in this case, and

$$J_1 = -\left(\frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2} - k_1\right)\phi_s.$$

Appendix B

When the Boussinesq approximation has been applied the equation for the perturbation vorticity is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\hat{\psi} - U''\hat{\psi}_x + \epsilon(\hat{\psi}_y\nabla^2\hat{\psi}_x - \hat{\psi}_x\nabla^2\hat{\psi}_y) - \frac{g}{\rho}\hat{\rho}_x = 0. \tag{B 1}$$

Equation (2.1) is obtained by eliminating $\hat{\rho}_x$ between (B 1) and (2.2). $\Phi_{20}(y,\tau)$ must satisfy (B 1), i.e.

$$\mu\Phi_{20yy\tau} = -F_2(y,\tau), \tag{B 2}$$

where

$$F_2(y,\tau) = -i[\Phi_{1y}\nabla_1^2\Phi_1^* - \Phi_{1y}^*\nabla_1^2\Phi_1 + \Phi_1\nabla_1^2\Phi_{1y}^* - \Phi_1^*\nabla_1^2\Phi_{1y}].$$

The equation for $\rho_{20}(y,\tau)$ is obtained from (2.2), i.e.

$$\mu\rho_{20\tau} = i[\Phi_{1y}\rho_1^* - \Phi_{1y}^*\rho_1 + \Phi_1\rho_{1y}^* - \Phi_1^*\rho_{1y}], \tag{B 3}$$

where ρ_1 is given by (2.7).

We introduce the expression for $\Phi_1(y,\tau)$ given by (2.10) into the expression for $F_2(y,\tau)$. In the cases in which we are interested, ϕ_s is a real function, and ϕ_{12} is purely imaginary. That ϕ_{12} is purely imaginary follows from (2.27), since $k_1 = 0$ in our case. Consequently $\phi_s^* = \phi_s$ and $\phi_{12}^* = -\phi_{12}$, which yields

$$\begin{aligned} F_2(y,\tau) &= -i\mu\frac{d}{d\tau}[AA^*][\phi_{12}\nabla_{10}^2\phi_s' - \phi_s'\nabla_{10}^2\phi_{12} + \phi_{12}'\nabla_{10}^2\phi_s - \phi_s\nabla_{10}^2\phi_{12}'] \\ &= -\mu\frac{d}{d\tau}[AA^*]\left[\left(-\frac{2\beta g}{(U-c_s)^3} + \frac{U''}{(U-c_s)^2}\right)\phi_s^2\right]', \end{aligned} \tag{B 4}$$

where we have used (2.16) and (2.24) for ϕ_s and ϕ_{12} and the fact that $k_1 = 0$.

Introducing the expression for $\Phi_{20}(y,\tau)$ given by (2.10) into (B 2) and using (B 4), we get

$$\phi_{201}'' = \left[\left(-\frac{2\beta g}{(U-c_s)^3} + \frac{U''}{(U-c_s)^2}\right)\phi_s^2\right]'. \tag{B 5}$$

ϕ_{201} given by (B 5) will have no singularity on $[y_1, y_2]$ for the flows studied in §§3 and 4 or for any other flows of the same class (cf. §2).

The right-hand side of (B 3) is found to be zero when we introduce ρ_1 given by (2.7) into it. Therefore

$$\rho_{201} = 0. \tag{B 6}$$

Equation (B 5) also gives Schade's result for the inviscid mean-flow distortion (Schade 1964, (16)). In Schade's case $U = \tanh y$, $\beta = 0$ and $\phi_s = \operatorname{sech} y$. It should be noted that in this case ϕ_{12} is not purely imaginary, because k_1 is now purely imaginary. However, we obtain (B 5) also in this case (with $\beta = 0$ of course), but in addition we get $A\, dA^*/d\tau = A^*\, dA/d\tau$. In Schade's case the mean-flow distortion given by (B 5) is singular at the critical layer, and he therefore rejected it. Instead he included the effect of the viscosity within the critical layer, and found that the mean-flow distortion had to be zero for infinite Reynolds number.

Maslowe (1977*b*) considered the mean-flow distortion for finite Reynolds numbers for the model studied by Schade. Recently Huerre (1980) studied the same model and found that the effect of the mean-flow distortion should not be neglected.

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